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# Diffusive representations for fractional Laplacian: systems theory framework and numerical issues

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**Abstract**
Bridging the gap between an abstract definition of pseudo-differential operators, such as $(-\Delta)^{\gamma}$ for $-\frac{1}{2} < \gamma < \frac{1}{2}$, and a *concrete* way to represent them has proved difficult; deriving stable numerical schemes for such operators is not an easy task either. Thus, the framework of *diffusive representations*, as already developed for causal fractional integrals and derivatives, is being applied to fractional Laplacian: it can be seen as an extension of the Wiener–Hopf factorization and splitting techniques to irrational transfer functions.

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## 1. Introduction

Fractional Laplacian has drawn research attention in the field of *electromagnetics* in the modeling of phenomena in bounded domains with an edge: the fractional order is then linked to the angle of the edge, see e.g. [1, 2][1] or the more recently published book on metamaterials [3]. It has also attracted attention in the modeling of damping in *continuum mechanics*, see e.g. [4, 5]. Finally, it has attracted the renewed interest of the *control theory* community, see e.g. [6] and the more recent works [7–9].

From the more mathematical side, many numerical techniques have been developed to treat this kind of pseudo-differential operator, see e.g. [10–13] to cite only a few; even though the theory of *Riesz potentials* is not new, and can be accounted for in e.g. sections 12 and 25 of [14], we find it quite difficult to bridge the gap between these abstract pseudo-differential operators and a *concrete* way of representing them, in a sense close to realization theory; even more difficult is the task of deriving stable numerical schemes for such systems.

The aim of this work, started in [15], is to show that elementary first-order systems, either causal or anti-causal, with appropriate aggregation, lead to the representation of *non-causal* pseudo-differential operators, such as

- $y = (-\Delta)^{-\beta/2}\, u$, called the Riesz fractional integral of order $0 < \beta < 1$,
- $z = (-\Delta)^{+\alpha/2}\, u$, called the Riesz fractional derivative of order $0 < \alpha < 1$.

The underlying ideas are those of *diffusive representations*, as already fully developed in section 5 of [16, 17], and recast in the framework of *well-posed systems* in [18]; these ideas have been applied to the examples of

- Riemann–Liouville *causal* fractional integrals $(\partial_t^{-\beta})$,
- fractional derivatives in the sense of *causal* distributions of Schwartz $(\partial_t^{\alpha})$;

these ideas are then combined with the Wiener–Hopf techniques, as detailed in e.g. section 7 of [19].

The paper is organized as follows: in section 2, we recall what fractional Laplacian means in various cases, in section 3 we give a primer on *causal* diffusive representation together with the link with fractional calculus, and then in section 4, we decompose the *tempered* solution of a parameterized ordinary differential equation (ODE) into its causal and anti-causal parts and aggregate them in order to get a realization of the fractional Laplacian; finally, in section 5, we list some interesting numerical consequences of the former diffusive representation, and in section 6 we enumerate a number of open questions, to be examined in the near future.

[1] The author thanks Professor D Bajon from ISAE (DEOS) for communicating this reference to him.

## 2. Definitions of fractional Laplacian

### *2.1. Fractional powers for matrices*

We recall the spectral theorem for symmetric real-valued matrices: if $A = A^{\mathrm{T}} \in M_{n\times n}(\mathbb{R})$, then there exists a diagonal matrix $\Lambda$ and an orthogonal matrix $P$ (i.e. $P^{\mathrm{T}} P = I_n$), such that $A = P^{-1} \Lambda P$. Then, for the fractional power of a symmetric matrix, two cases may occur:

(i) if $A = A^{\mathrm{T}} > 0$, i.e. $A$ is positive definite, then one can uniquely define $A^{-\beta} = P^{-1} \Lambda^{-\beta} P$, with $\Lambda^{-\beta} = \mathrm{diag}(\lambda_1^{-\beta}, \ldots, \lambda_n^{-\beta})$, since $\lambda_i > 0$.
(ii) if $A = A^{\mathrm{T}} \geqslant 0$, i.e. $A$ is positive, then one can uniquely define $A^{\alpha} = P^{-1} \Lambda^{\alpha} P$, with $\Lambda^{\alpha} = \mathrm{diag}(\lambda_1^{\alpha}, \ldots, \lambda_n^{\alpha})$, since $\lambda_i \geqslant 0$.

### *2.2. Fractional powers for operators with a compactness property*

In an infinite-dimensional setting, things are much more complicated: we begin with the case of *bounded* operators.

*2.2.1. Bounded operators.* Following e.g. [9], if $K$ is a *compact* and symmetric operator on a Hilbert space $\mathcal{H}$, using the spectral theorem, we get a spectral mapping theorem of the above form:

(i) if $K = K^{\mathrm{T}} > 0$, i.e. $K$ is positive definite, then one can uniquely define $K^{-\beta} = P^{-1} \Lambda^{-\beta} P$, with $\Lambda^{-\beta} = \mathrm{diag}(k_n^{-\beta})_{n\in\mathbb{N}}$, since $k_n > 0$; this unbounded operator is defined on a domain $D(K^{-\beta})$, see section 2.2.2.
(ii) if $K = K^{\mathrm{T}} \geqslant 0$, i.e. $K$ is positive, then one can uniquely define $K^{\alpha} = P^{-1} \Lambda^{\alpha} P$, with $\Lambda^{\alpha} = \mathrm{diag}(k_n^{\alpha})_{n\in\mathbb{N}}$, since $k_n \geqslant 0$.

The transform $P$ is unitary on $\mathcal{H}$, and the eigenvalues of $K$ consist of a sequence of positive real numbers $k_n$ that converge towards 0.

*2.2.2. Unbounded operators.* Now if $A$ is unbounded on $\mathcal{H}$, with a dense domain $D(A)$ in $\mathcal{H}$, self-adjoint, positive, and has *compact resolvent*, then the previous setting can be applied to $K_\rho = (\rho I - A)^{-1}$ for $\rho \in \rho(A)$, the resolvent set of $A$; in particular the eigenvalues of $A$ form a discrete sequence $\lambda_n$ of positive real numbers, which grows towards infinity. When $\lambda = 0$ is not an eigenvalue of $K_\rho$, the eigenvectors $(e_n)_{n\in\mathbb{N}}$ of $K_\rho$ form a Hilbert basis of $\mathcal{H}$, and we get the spectral theorem:

$$\forall \varphi \in \mathcal{H}, \quad \varphi = \sum_{n\in\mathbb{N}} (\varphi,\, e_n)\, e_n,$$

with the energy identity $\|\varphi\|_{\mathcal{H}}^2 = \sum_{n\in\mathbb{N}} |(\varphi,\, e_n)|^2$. And for any $\gamma > 0$, we can define the fractional power of $A$ as follows:

$$A^{\gamma}\varphi = \sum_{n\in\mathbb{N}} {\lambda_n}^{\gamma} (\varphi,\, e_n)\, e_n,$$

provided $\varphi \in D(A^{\gamma})$, where

$$D(A^{\gamma}) = \left\{ \varphi \in \mathcal{H},\ \sum_{n\in\mathbb{N}} {\lambda_n}^{2\gamma}\, |(e_n,\, \varphi)|^2 < \infty \right\}.$$

This is indeed the case for $A = -\partial_{x^2}^2$ on the *bounded* interval $I = (0,\, 1)$, with the Dirichlet (D) or Neumann (N) boundary conditions at each end:

- D–D case: $\lambda_n = n^2\pi^2$ and $e_n \propto \sin(n\pi x)$ for $n \geqslant 1$,
- D–N case: $\lambda_n = (n + \frac{1}{2})^2\pi^2$ and $e_n \propto \sin((n + \frac{1}{2})\pi x)$ for $n \geqslant 0$,
- N–D case: $\lambda_n = (n + \frac{1}{2})^2\pi^2$ and $e_n \propto \cos((n + \frac{1}{2})\pi x)$ for $n \geqslant 0$,
- N–N case: $\lambda_n = n^2\pi^2$ and $e_n \propto \cos(n\pi x)$ for $n \geqslant 0$ (note that $\lambda = 0$ is indeed an eigenvalue).

Also useful is the case of $A_{\mathrm{per}} = -\partial_{x^2}^2$ with *periodic* boundary conditions on $I$, leading to

(i) $\lambda_0 = 0$ and $e_0 = 1$,
(ii) for $n \geqslant 1$, $\lambda_n = 4\pi^2 n^2$, the two-dimensional (2D) eigenspace being spanned by orthogonal eigenvectors $e_{n,1} \propto \cos(2\pi n x)$ and $e_{n,2} \propto \sin(2\pi n x)$;

in which case we recover the celebrated *Fourier series* decomposition.

### *2.3. Fractional powers for operators without a compactness property*

Consider $A = -\partial_{x^2}^2$ on the unbounded interval $\mathbb{R}$, we know from Fourier analysis in $\mathcal{H} = L^2(\mathbb{R})$ that this operator can be diagonalized as follows, with $P = \mathcal{F} : L^2(\mathbb{R}_x) \to L^2(\mathbb{R}_\xi)$, the unitary Fourier transform, and $P^{-1} = \mathcal{F}^{-1} : L^2(\mathbb{R}_\xi) \to L^2(\mathbb{R}_x)$:

$$\begin{aligned} \hat{A} : L^2(\mathbb{R}_\xi) &\to L^2(\mathbb{R}_\xi), \\ \hat{\varphi} &\mapsto 4\pi^2\xi^2\, \hat{\varphi}\,, \end{aligned} \tag{1}$$

on the domain $D(\hat{A}) = L^{2,2}(\mathbb{R}_\xi)$, where we have set

$$L^{2,s}(\mathbb{R}_\xi) := \left\{ \hat{\varphi} \in L^2(\mathbb{R}_\xi),\ \int_{\mathbb{R}} (1 + 4\pi^2\xi^2)^s\, |\hat{\varphi}|^2\, \mathrm{d}\xi < \infty \right\}.$$

For $\gamma > 0$, it is then not difficult to define the fractional power of $A$ in the following way: $A^{\gamma} = P^{-1}\, \hat{A}^{\gamma}\, P$, where

$$\begin{aligned} \hat{A}^{\gamma} : L^2(\mathbb{R}_\xi) &\to L^2(\mathbb{R}_\xi) \\ \hat{\varphi} &\mapsto (4\pi^2\xi^2)^{\gamma}\, \hat{\varphi}\,, \end{aligned} \tag{2}$$

on the domain $D(\hat{A}^{\gamma}) = L^{2,2\gamma}(\mathbb{R}_\xi)$. We can see here that since the compactness is lost, no Hilbert basis will help diagonalize the operator; even though the space $L^2(\mathbb{R}_x)$ is separable, meaning it has a countable family that is everywhere dense, such as the Hermite functions.

### *2.4. Questions*

All these formulae are quite nice and could be computed, at least *theoretically*, but

(i) it appears clearly that the input $\varphi$ must *first* be represented on the eigenfamily (*analysis* step), either by $(\varphi, e_n)_{n\in\mathbb{N}}$ or by $(\hat{\varphi}(\xi))_{\xi\in\mathbb{R}}$, which is not an easy task in general;
(ii) then, in a second stage, the result of $A^{\gamma}\varphi$ is built by a series or an integral (*synthesis* step).

The aim of the present work is to show that the same result can also be recovered by the output of a *dynamical* system, thus allowing for online treatment of the fractional power operator, and even some real-time computations, see e.g. [20].

## 3. A primer on diffusive representations and fractional calculus

In this section, we focus on the *causal* solution of a family of first-order ODEs. Hence, the mathematical setting is the convolution algebra $\mathcal{D}'_+(\mathbb{R})$ of causal distributions.

### 3.1. An elementary approach

Consider the numerical identity, valid for $\delta > 1$:

$$\int_0^\infty \frac{\mathrm{d}x}{1+x^\delta} = \frac{\frac{\pi}{\delta}}{\sin(\pi/\delta)}\,.$$

Letting $s \in \mathbb{R}^+_*$ and substituting $x = (\frac{\xi}{s})^{\frac{1}{\delta}}$ in the above *numerical* identity, we get

$$\int_0^\infty \frac{\sin(\pi/\delta)}{\pi}\,\frac{1}{\xi^{1-\frac{1}{\delta}}}\,\frac{1}{s+\xi}\,\mathrm{d}\xi = \frac{1}{s^{1-\frac{1}{\delta}}}.$$

Finally, performing an analytic continuation from $\mathbb{R}^{+*}$ to $\mathbb{C}\setminus\mathbb{R}^-$ for both sides of the above identity in the complex variable $s$, and letting $\beta := 1-\frac{1}{\delta} \in (0,\,1)$, we get the *functional* identity:

$$\begin{aligned} H_\beta &: \mathbb{C}\setminus\mathbb{R}^- \to \mathbb{C} \\ &s \mapsto \int_0^\infty \mu_\beta(\xi)\,\frac{1}{s+\xi}\,\mathrm{d}\xi = \frac{1}{s^\beta}\,, \end{aligned} \tag{3}$$

with density $\mu_\beta(\xi) = \frac{\sin(\beta\,\pi)}{\pi}\,\xi^{-\beta}$.

Applying an inverse Laplace transform to both sides gives

$$\begin{aligned} h_\beta &: \mathbb{R}^+ \to \mathbb{R} \\ &t \mapsto \int_0^\infty \mu_\beta(\xi)\,\mathrm{e}^{-\xi t}\,\mathrm{d}\xi = \frac{1}{\Gamma(\beta)}t^{\beta-1}\,. \end{aligned} \tag{4}$$

### 3.2. Input–output representation

Let $u$ and $y = I^\beta u$ be the input and output of the *causal* fractional integral of order $\beta$, defined by the Riemann–Liouville formula $y = h_\beta * u = \int_0^t h_\beta(t-\tau)\,u(\tau)\,\mathrm{d}\tau$ in the time domain, which reads $Y(s) = H_\beta(s)\,U(s)$ in the Laplace domain.

Using the integral representations above, together with Fubini's theorem, we get

$$y(t) = \int_0^\infty \mu_\beta(\xi)\,[\mathrm{e}_\xi * u](t)\,\mathrm{d}\xi$$

with $e_\xi(t) := \mathrm{e}^{-\xi\,t}$, and $[e_\xi * u](t) = \int_0^t \mathrm{e}^{-\xi\,(t-\tau)}\,u(\tau)\,\mathrm{d}\tau$.

Now for a fractional *derivative* of order $\alpha \in (0,\,1)$ in the sense of the distributions of Schwartz, we have $z = \mathrm{D}^\alpha u = \mathrm{D}[I^{1-\alpha} u]$, and a careful computation shows that

$$z(t) = \int_0^\infty \mu_{1-\alpha}(\xi)\,[u - \xi\,e_\xi * u](t)\,\mathrm{d}\xi.$$

### 3.3. State space representation

In both the above input–output representations, introducing a state, say $\varphi(\xi,.)$, that realizes the classical convolution $\varphi(\xi,.) := [e_\xi * u](t)$ leads to the following diffusive *realizations*, in the sense of systems theory:

$$\partial_t\varphi(\xi,t) = -\xi\,\varphi(\xi,t) + u(t), \quad \varphi(\xi,0) = 0, \tag{5}$$

$$y(t) = \int_0^\infty \mu_\beta(\xi)\,\varphi(\xi,t)\,\mathrm{d}\xi; \tag{6}$$

and

$$\partial_t\widetilde{\varphi}(\xi,t) = -\xi\,\widetilde{\varphi}(\xi,t) + u(t), \quad \widetilde{\varphi}(\xi,0) = 0, \tag{7}$$

$$z(t) = \int_0^\infty \mu_{1-\alpha}(\xi)\,[u(t) - \xi\,\widetilde{\varphi}(\xi,t)]\,\mathrm{d}\xi. \tag{8}$$

These are first and extended diffusive realizations, respectively. The slight difference between (5) and (7), marked by the notation $^\sim$, lies in the underlying functional spaces in which these equations make sense: $\varphi$ belongs to $\mathcal{H}_\beta := \{\varphi \text{ s.t. } \int_0^\infty \mu_\beta(\xi)|\varphi|^2\,\mathrm{d}\xi < \infty\}$, whereas $\widetilde{\varphi}$ belongs to $\widetilde{\mathcal{H}}_\alpha := \{\widetilde{\varphi} \text{ s.t. } \int_0^\infty \mu_{1-\alpha}(\xi)|\widetilde{\varphi}|^2\,\xi\,\mathrm{d}\xi < \infty\}$, see e.g. [18].

## 4. Tempered solutions of ODEs

In this section, we focus on the *tempered* solution of a family of first-order ODEs. Hence, the mathematical setting is the space $\mathcal{S}'(\mathbb{R})$ of tempered distributions.

### 4.1. A second order ODE

Let $\lambda > 0$, and consider the second order ODE $y'' - \lambda^2\,y = u$, it has at least two very different types of solutions:

(i) the *causal* solution of $y'' - \lambda^2\,y = u$ in $\mathcal{D}'_+(\mathbb{R})$ is

$$y = h_c * u, \quad \text{with } h_c := \frac{1}{\lambda}\sinh(\lambda\,x)\,1_{x\geqslant 0},$$

(ii) the *tempered* solution of $y'' - \lambda^2\,y = u$ in $\mathcal{S}'(\mathbb{R})$ is

$$y = h_s * u, \quad \text{with } h_s(x) := -\frac{1}{2\lambda}\mathrm{e}^{-\lambda\,|x|}.$$

Understanding this main difference is of the utmost importance to generalize diffusive representations to the tempered distribution framework.

Following section 7.1 of [19], the stable kernel $h(x) = 2\,\mathrm{e}^{-|x|}$ can be decomposed into $h^\pm$, with support $\mathbb{R}^\pm$; hence the convolution $y = h * u$ can be seen as the sum of two subsystems, namely $\mp\partial_x\varphi^\pm(x) = -1\,\varphi^\pm(x) + u(x)$, $\varphi^\pm(0) = 0$, and $y = \varphi^+ + \varphi^-$.

In the next section, we apply this decomposition to $h_\beta(x) = \frac{1}{\Gamma(\beta)}\,|x|^{\beta-1}$, using the diffusive realization of some irrational transfer functions with branchpoints (see section 7.2 of [19]).

#### 4.2. Decomposition into causal and anti-causal first-order ODEs

Let $\phi^+(\lambda, x)$ be the *causal* solution of

$$\begin{aligned} &\partial_x \phi^+(\lambda, x) = -\lambda\, \phi^+(\lambda, x) + u(x), \quad \lambda > 0, \\ &\phi^+(\lambda, x=0) = 0, \quad \lambda > 0, \end{aligned} \tag{9}$$

and define the standard and extended observations as

$$\begin{aligned} y^+(x) &= \int_0^\infty \phi^+(\lambda, x)\, \mu_\beta(\lambda)\, \mathrm{d}\lambda, \\ z^+(x) &= \int_0^\infty \left[-\lambda\, \phi^+(\lambda, x) + u(x)\right] \mu_\beta(\lambda)\, \mathrm{d}\lambda. \end{aligned} \tag{10}$$

Now let $\phi^-(\lambda, x)$ be the *anti-causal* solution of

$$\begin{aligned} &-\partial_x \phi^-(\lambda, x) = -\lambda\, \phi^-(\lambda, x) + u(x), \quad \lambda > 0, \\ &\phi^-(x=0) = 0, \quad \lambda > 0, \end{aligned} \tag{11}$$

and define the standard and extended observations as

$$\begin{aligned} y^-(x) &= \int_0^\infty \phi^-(\lambda, x)\, \mu_\beta(\lambda)\, \mathrm{d}\lambda, \\ z^-(x) &= \int_0^\infty \left[-\lambda\, \phi^-(\lambda, x) + u(x)\right] \mu_\beta(\lambda)\, \mathrm{d}\lambda. \end{aligned} \tag{12}$$

#### 4.3. Linear aggregation of tempered solutions

Then, for the particular choice $\mu_\beta(\lambda) = \frac{\sin \beta\pi}{\pi} \lambda^{-\beta}$, we get the following results

(i) the *standard* output

$$y := \frac{1}{2\cos(2\beta\pi)} \left[y^+ + y^-\right] = (-\Delta)^{-\beta/2} u,$$

(ii) whereas the *extended* output

$$z := \frac{1}{2\cos(2\alpha\pi)} \left[z^+ + z^-\right] = (-\Delta)^{+\alpha/2} u,$$

with $\alpha := 1 - \beta$, as usual.

The proof is straightforward and left to the reader.

Thus, we have performed the representation of the fractional Laplacian of negative or positive order in dimension $N = 1$ in a very simple way: it is the combination of two diffusive realizations, the first is causal, whereas the second is anti-causal; for negative powers $\gamma = -\beta/2 \in (-1/2, 0)$, *standard* diffusive realizations are being used, whereas for positive powers $\gamma = +\alpha/2 \in (0, 1/2)$, *extended* diffusive realizations are being used.

**Remark 1.** As for the causal case, useful *energy balances* w.r.t. continuous variables could be established, which should prove useful for both the analysis of coupled systems and stability analysis.

#### 4.4. Tackling multidimensional cases

As far as we know, diffusive representations for 2D problems have been first used in [21], and then in [22], but it must be noticed that always *causal* representations have been used.

Using section 25 of [14], the Riesz fractional integro-differentiation in $\mathbb{R}^N$ $(N \geqslant 2)$ can be recast in the previous framework, at least for *radially symmetric* inputs; in which case, only causal diffusive representations would be used, with respect to the radius variable $r \in (0, \infty)$.

Otherwise, we believe that our approach could still be followed, and would then give rise to averaged and parameterized diffusive representations. Let us make ourselves more explicit in the case $N = 2$. Let $u(x_1, x_2) := \check{u}(r, \theta)$ with straightforward notations, and consider for $\theta \in (-\pi, \pi)$:

$$\partial_r \varphi(\lambda, r, \theta) = -\lambda\, \varphi(\lambda, r, \theta) + \check{u}(r, \theta), \quad \lambda > 0, \quad \varphi(r{=}0) = 0,$$

$$\check{y}(r, \theta) = \int_0^\infty \varphi(\lambda, r, \theta)\, \mu_\beta(\lambda)\, \mathrm{d}\lambda,$$

$$\check{z}(r, \theta) = \int_0^\infty \left[-\lambda\, \varphi(\lambda, r, \theta) + \check{u}(r, \theta)\right] \mu_{1-\beta}(\lambda)\, \mathrm{d}\lambda,$$

from which we can easily recover $y(x_1, x_2)$ and $z(x_1, x_2)$. So far, the question is left open to know whether the pseudo-differential operator that has been realized in this way is precisely the fractional Laplacian, or only the convolution by the fractional Riesz potential: this point has to be clarified; some averaging w.r.t. the $\theta$ variable should perhaps be performed in the last stage.

**Remark 2.** Funnily enough, in the much simpler case of dimension $N = 1$, parameter $\theta$ lies in $\{-1, 1\}$; that is why the so-called parameterized diffusive representations can be seen or interpreted as *two* families of diffusive representations only, the causal one for $\theta = 1$ and the anti-causal one for $\theta = -1$.

## 5. Consequences on numerical simulation

We only list some of the many available choices for the numerical counterpart of these integral representations:

(i) Interpolation: see e.g. [20],
(ii) Optimization: see [23] and the more recent paper [20],
(iii) Efficient decomposition for fast computation: see [24] and references therein,
(iv) Spectral order methods: see e.g. [25] and references therein.

**Remark 3.** The energy balances established w.r.t. the continuous variables will have to be derived at the discrete level; they will prove useful in order to ensure the stability of the numerical schemes.

## 6. Conclusions and future works

#### 6.1. Conclusions

What has been done so far lies in the simplification and decomposition of fractional Laplacian into elementary first-order systems, with tempered solution (either causal or anti-causal) and a specific aggregation of these systems: it clearly illustrates the interplay between mathematics, systems theory and numerical analysis. In a way, it makes weird pseudo-differential operators much more concrete and not that difficult to apprehend, either on the theoretical level or on the numerical level.

The multidimensional case (section 4.4) and the possible numerical consequences of this approach (section 5) will need to be further developed; they should provide straightforward results of numerical experiments in the following two cases:

- $(-\partial_{xx}^2)^\gamma$ on the real line,
- $(-\Delta)^\gamma$ on the whole space of dimension 2, for radially symmetric inputs.

### 6.2. An alternative approach

Note that another way of getting more concrete representations of fractional powers of Laplacian is as follows: many mathematically oriented works, such as [9], first identify a positive self-adjoint operator $\mathcal{A}$, and define in an abstract way fractional powers of this operator, then some damping is introduced through $\mathcal{A}^{\frac{1}{2}}$ for example, maybe in order to preserve the modal structure, if any. But, for sure, one of the problems of this approach is that it is quite formal and proves far away from physical modeling: the first clue is that $\mathcal{A}^{\frac{1}{2}}$ is never a differential operator, as already seen.

On the contrary, a factorization of the *Cholevsky* type could be more useful: $-\Delta = -\mathrm{div}\,\mathbf{grad}$ with $\mathbf{grad}^{\mathrm{T}} = -\mathrm{div}$ involves two differential operators, the former being the adjoint of the latter on $L^2$, which actually preserves positivity, since $(u, -\Delta u) = (u, \mathbf{grad}^{\mathrm{T}}\, \mathbf{grad}\, u) = \| \mathbf{grad}\, u \|^2 \geqslant 0$.

This way of factorizing things out also has the most interesting *numerical* consequences: in dimension $N = 1$, discretizing $-\Delta u$ by $-Du_n := -u_{n+1} + 2u_n - u_{n-1}$, the symbol of which is $-z + 2 - z^{-1}$, with discrete Fourier transform $2(1 - \cos\omega) = (2\sin\omega/2)^2$ does preserve the *positivity* property $(u, -Du) \geqslant 0$ for the scalar product in $l^2(\mathbb{Z})$. In fact, a very nice way to see and prove this is to perform the self-adjoint factorization as follows: $-D = (I - B)(I - B^{\mathrm{T}})$, where $B$ is the delay or backward difference operator and $B^{\mathrm{T}}$ is the forward difference operator; hence $(u, -Du) = (u, (I - B^{\mathrm{T}})(I - B)u) = \| (I - B)u \|^2 \geqslant 0$.

Hence, this alternative approach, which will be fully carried out in [5], proves effective at least in the case $\gamma = \frac{1}{2}$.

### 6.3. Open questions and possible extensions

The energy balances must be carefully derived, both for the causal and the anti-causal parts; then, they must be defined at the discrete level also.

At this stage, it is not so clear how to use the previous setting on $(-\partial_{x^2}^2)^\gamma$ in a bounded domain, without the periodicity assumption: how can we use the framework above in this case?

For the analysis of a beam, it is the bi-Laplacian operator that is involved; hence fractional powers such as $(\partial_{x^4}^4)^\gamma$ will be interesting in this case, with $\gamma \in (-1/2,\ 1/2)$, and appropriate boundary conditions: how can we extend the work done on $(-\partial_{x^2}^2)^\gamma$ in a bounded domain to this new operator? Note that in this case, $(\partial_{x^4}^4)^{1/2} \neq -\partial_{x^2}^2$ in general.

To what extent is the representation useful to analyze closed-loop systems and their numerical counterparts, since the diffusive dynamical system with respect to some space variable is not causal?

Can other pseudo-differential operators ($\psi$DO) be represented in the same way, such as $(-\partial_x[a(x)\partial_x])^\gamma$ in 1D with $a \geqslant 0$?

Can nonlinear aggregation be investigated also, instead of linear, as presented in section 4.3? What are the conditions on the nonlinearity, in order to extend some non-quadratic energy balances? Has this kind of work already been done somewhere else?

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## References

[1] Altman Z, Renaud D and Baudrand H 1994 On the use of differential equations of nonentire order to generate entire domain basis functions with edge singularity *IEEE Trans. Microw. Theory Tech.* **42** 1966–72
[2] Engheta N 2000 Fractional paradigm in electromagnetic theory *Frontiers in Electromagnetics* (Piscataway, NJ: IEEE) chapter 12, pp 523–52
[3] Engheta N and Ziolkowski R W (eds) 2006 *Metamaterials: Physics and Engineering Explorations* (Piscataway, NJ: IEEE)
[4] Adhikari S 2006 Damping modelling using generalized proportional damping *J. Sound Vib.* **293** 156–70
[5] Matignon D and Hélie Th 2009 Damping models of linear waves for the sound synthesis of vibrating structures (in preparation)
[6] Hansen S 2000 Optimal regularity results for boundary control of elastic systems with fractional order damping *ESAIM: Proc.* **8** 53–64
[7] Miller L 2006 On the controllability of anomalous diffusions generated by the fractional Laplacian *Math. Control Signals Syst.* **18** 260–71
[8] Micu S and Zuazua E 2006 On the controllability of a fractional order parabolic equation *SIAM J. Control Optim.* **44** 1950–72
[9] Tucsnak M and Weiss G 2008 *Observability and Controllability of Infinite-Dimensional Systems* http://www.ee.ic.ac.uk/gweiss/personal/obs_book.pdf
[10] Roop J P 2004 Variational solution of the fractional advection dispersion equation *PhD Thesis* Clemson University
[11] Ervin V J, Heuer N and Roop J P 2007 Numerical approximation of a time dependent, non-linear, fractional order diffusion equation *SIAM J. Numer. Anal.* **45** 572–91
[12] Roop J P 2006 Computational aspects of FEM approximation of fractional advection dispersion equations on bounded domains in $\mathbb{R}^2$ *J. Comput. Appl. Math.* **193** 243–68
[13] Ervin V J and Roop J P 2006 Variational formulation for the stationary fractional advection dispersion equation *Numer. Methods Partial Differ. Equations* **22** 558–76
[14] Samko S G, Kilbas A A and Marichev O I 1993 *Fractional Integrals and Derivatives: Theory and Applications* (Gordon and Breach) (Transl. from the Russian) (Originally publ. 1987)

[15] Matignon D 2007 Diffusive representation for fractional Laplacian and other non-causal pseudo-differential operators *Workshop on Control of Distributed Parameter Systems* (*Namur, Belgium, July 2007*) (IFAC) pp 19–20

[16] Staffans O J 1994 Well-posedness and stabilizability of a viscoelastic equation in energy space *Trans. Am. Math. Soc.* **345** 527–75

[17] Matignon D and Montseny G (eds) 1998 *Fractional Differential Systems: Models, Methods and Applications (ESAIM: Proceedings* vol 5) (SMAI)

[18] Matignon D and Zwart H 2008 Standard diffusive systems as well-posed linear systems (in preparation)

[19] Duffy D G 1994 *Transform Methods for Solving Partial Differential Equations* (Boca Raton, FL: CRC Press)

[20] Hélie Th and Matignon D 2006 Representations with poles and cuts for the time-domain simulation of fractional systems and irrational transfer functions *Signal Process.* **86** 2516–28

[21] Audounet J, Matignon D and Montseny G 2000 Perfectly absorbing boundary feedback control for wave equations: a diffusive formulation *Fifth Int. Conf. on Mathematical and Numerical Aspects of Wave Propagation Phenomena* (*Santiago de Compostela, Spain, July 2000*) (INRIA, SIAM) pp 1025–9

[22] Levadoux D and Montseny G 2003 Diffusive realization of the impedance operator on circular boundary for 2D wave equation *Sixth Int. Conf. on Mathematical and Numerical Aspects of Wave Propagation Phenomena* (*Jyväskylä, Finland, July 2003*) (INRIA) pp 136–41

[23] Garcia G and Bernussou J 1998 Identification of the dynamics of a lead acid battery by a diffusive model *ESAIM: Proc.* **5** 87–98

[24] Haddar H, Li J-R and Matignon D 2009 Efficient solution of a wave equation with fractional order dissipative terms *J. Comput. Appl. Math.* at press

[25] Weideman J A C and Trefethen L N 2007 Parabolic and hyperbolic contours for computing the Bromwich integral *Math. Comput.* **76** 1341–56